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EVOLUTION OF LONG NONLINEAR WAVES ON THE INTERFACE OF A STRATIFIED VISCOUS FLUID FLOW IN A CHANNEL

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The dynamics of disturbances of the interface between two layers of incompressible immiscible fluids of different densities in the presence of a steady flow between the horizontal bottom and lid is studied analytically and numerically. A model integrodifferential equation is derived, which takes into account long-wave contributions of inertial layers and surface tension of the fluids, small but finite amplitude of disturbances, and unsteady shear stresses on all boundaries. Numerical solutions of this equation are given for the most typical nonlinear problems of transformation of both plane waves of different lengths and solitary waves.

Key words: *viscous fluids, interface, two-layer Poiseuille flow, long waves, nonlinear disturbances, solitary waves.*

Introduction. Disturbances on free surfaces of shallow fluid layers with a shear in streamwise velocity have been of interest for specialists for half a century (see, e.g., [1]). Attention to research of this kind has become even more intense recently (see [2, 3] and other papers). In particular, Poloukhin et al. [4] performed measurements under natural conditions and studied the effect of shear flows on the vertical structure and kinematic parameters of internal waves. Arkhipov and Khabakhpashev [5] derived an evolutionary equation for plane nonlinear disturbances of the interface of a two-layer Poiseuille flow. In contrast to other models, this equation takes into account unsteady shear stresses on all boundaries of the system. It was found that the flow velocity and direction can alter not only the wavelength but also the wave polarity. Dissipative losses being neglected, steady-state solutions of the type of cnoidal and solitary waves were determined for a disturbed flow.

The present work is aimed at deriving a model equation for three-dimensional disturbances and analyzing numerical experiments on transformation of various waves.

Formulation of the Problem and Simplification of Initial Equations. Let the fluids be bounded by a rigid motionless lid (vertical coordinate $z = h_1$) and a rigid motionless bottom ($z = -h_2$), and the undisturbed interface between the layers correspond to the coordinate $z = 0$. Then, the steady-state profile of horizontal velocity consists of two segments of parabolas:

$$\boldsymbol{u}_{0l} = \boldsymbol{u}_{0i}(1 - A_l z^2 - B_l z),$$
$$\boldsymbol{u}_{0i} = -\frac{\nabla p_0 h_1 h_2 H/2}{\mu_1 h_2 + \mu_2 h_1}, \qquad A_l = \frac{\mu_1 h_2 + \mu_2 h_1}{\mu_l h_1 h_2 H}, \qquad B_l = \frac{\mu_1 h_2^2 - \mu_2 h_1^2}{\mu_l h_1 h_2 H}. \tag{1}$$

Here the gradient operator ∇ is determined in the horizontal plane (x, y), p is the pressure, $H = h_1 + h_2$, and μ is the dynamic viscosity; the subscript 0 refers to steady-state values of the quantities, the subscripts $l = 1$ and 2 refer to the upper and lower layers of the fluid, respectively, the subscript i refers to quantities on the interface.

The two-layer Poiseuille flow in a plane channel under consideration is the solution of the steady-state equation of motion $\nabla p_0 = \mu_l\, d^2\boldsymbol{u}_{0l}/dz^2$ with boundary conditions $\boldsymbol{u}_{0l} = 0$ for $z = -(-1)^l h_l$, $\boldsymbol{u}_{0l} = \boldsymbol{u}_{0i}$ and

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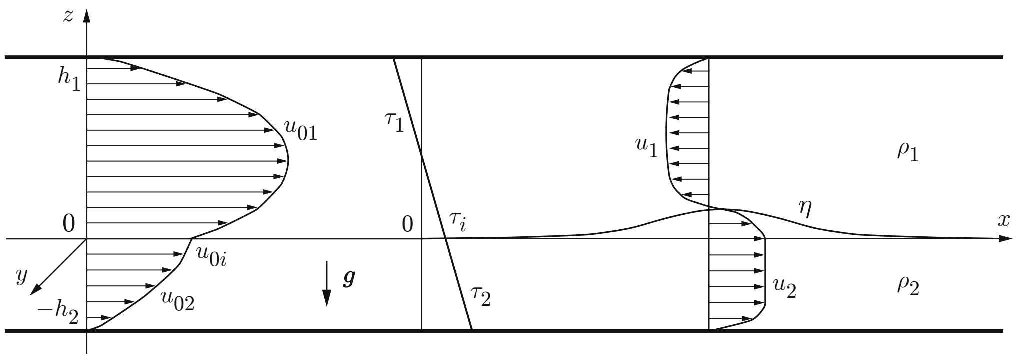

Fig. 1. Sketch of a steady horizontal flow, shear profile, and disturbed flow caused by propagation of a long wave in a two-layer system.

$\boldsymbol{\tau}_l = \mu_l \, d\boldsymbol{u}_{0l}/dz = \boldsymbol{\tau}_{0i}$ for $z = 0$. Figure 1 shows the profile of such a flow for a water–aniline system ($\mu_1/\mu_2 = 0.225$) with $h_1/h_2 = 2$.

The original Stokes equations and continuity equations for disturbed motion of an incompressible fluid in each layer can be written in a standard form as

$$\frac{\partial \boldsymbol{u}_l}{\partial t} + \boldsymbol{u}_{0l} \cdot \nabla \boldsymbol{u}_l + w_l \frac{d\boldsymbol{u}_{0l}}{dz} + \boldsymbol{u}_l \cdot \nabla \boldsymbol{u}_l + w_l \frac{\partial \boldsymbol{u}_l}{\partial z} + \frac{1}{\rho_l} \nabla p_l = \nu_l \Big(\nabla^2 \boldsymbol{u}_l + \frac{\partial^2 \boldsymbol{u}_l}{\partial z^2} \Big); \tag{2}$$

$$\frac{\partial w_l}{\partial t} + \boldsymbol{u}_{0l} \cdot \nabla w_l + \boldsymbol{u}_l \cdot \nabla w_l + w_l \frac{\partial w_l}{\partial z} + \frac{1}{\rho_l} \frac{\partial p_l}{\partial z} + g = \nu_l \Big(\nabla^2 w_l + \frac{\partial^2 w_l}{\partial z^2} \Big); \tag{3}$$

$$\nabla \cdot \boldsymbol{u}_l + \frac{\partial w_l}{\partial z} = 0. \tag{4}$$

Here $\boldsymbol{u}$ is the horizontal component of the velocity vector of the fluid, t is the time, w is the vertical component of the fluid velocity, ρ is the density, $\nu = \mu/\rho$ is the kinematic viscosity, and g is the acceleration of gravity.

Let us make the following assumptions: 1) the characteristic horizontal size of the wave l_w is substantially greater and the disturbance amplitude η_a is substantially smaller than the equilibrium depths of the layers h_l ($h_l/l_w \sim \varepsilon^{1/2}$ and $\eta_a/h_l \sim \varepsilon$, where ε is a small parameter); 2) the capillary effects are small [modified Bond number $\mathrm{Bo} = (\rho_2 - \rho_1) g h_1 h_2/\sigma > 1$, where σ is the surface tension]; 3) the boundary-layer thickness for disturbed velocity remains small, i.e., the time of propagation of the unsteady boundary layer over the fluid thickness is much greater than the characteristic time of wave propagation over a certain point of the examined region of the channel t_w (numbers of hydrodynamic homochromity $\mathrm{Ho}_{\nu l} = \nu_l t_w/h_l^2 \sim \varepsilon^2$). These assumptions correspond to test conditions in various hydrodynamic laboratories.

The nonlinear terms in Eqs. (3) can be omitted as terms of negligible order of smallness ($\boldsymbol{u}_l \cdot \nabla w_l \approx g\varepsilon^3$ and $w_l^2/\boldsymbol{u}_l^2 \approx \varepsilon$). Moreover, in the assumptions made, the first terms in the right sides of Eqs. (2) and the right sides of Eqs. (3) can also be neglected. As a result, the following simplified system is obtained:

$$\frac{\partial \boldsymbol{u}_l}{\partial t} + \boldsymbol{u}_{0l} \cdot \nabla \boldsymbol{u}_l + w_l \frac{d\boldsymbol{u}_{0l}}{dz} + \boldsymbol{u}_l \cdot \nabla \boldsymbol{u}_l + w_l \frac{\partial \boldsymbol{u}_l}{\partial z} + \frac{1}{\rho_l} \nabla p_l = \nu_l \frac{\partial^2 \boldsymbol{u}_l}{\partial z^2}; \tag{5}$$

$$\frac{\partial w_l}{\partial t} + \boldsymbol{u}_{0l} \cdot \nabla w_l + \frac{1}{\rho_l} \frac{\partial p_l}{\partial z} + g = 0. \tag{6}$$

The following boundary conditions are imposed on the lid, bottom, and interface between the layers:

$$\boldsymbol{u}_l = w_l = 0, \qquad z = -(-1)^l h_l,$$

$$\boldsymbol{u}_1 = \boldsymbol{u}_2 = \boldsymbol{u}_i, \quad \mu_1 \frac{\partial \boldsymbol{u}_1}{\partial z} = \mu_2 \frac{\partial \boldsymbol{u}_2}{\partial z} = \boldsymbol{\tau}_i, \quad w_1 = w_2 = w_i = \frac{\partial \eta}{\partial t} + (\boldsymbol{u}_{0i} + \boldsymbol{u}_i) \cdot \nabla \eta,$$

$$p_{1i} = p_{2i} + \sigma \nabla^2 \eta, \qquad z = \eta(t, x, y).$$

Integrating Eqs. (6) with respect to z from z to η and using the dynamic boundary condition on the interface, we find the pressure profiles in each layer of the fluid:

$$\frac{p_l}{\rho_l} = \frac{p_{li}}{\rho_l} + g(\eta - z) + \int\limits_z^\eta \Big(\frac{\partial w_l}{\partial t} + \boldsymbol{u}_{0l} \cdot \nabla w_l\Big)\, dz. \tag{7}$$

Expressions (7) can be substituted into the equations of motion (5), (6), but first it is reasonable to take integrals in dependence (7). It was shown [5] that, in a wide interval of depth ratios and moderate values of the steady-state hydraulic pressure, the profiles of the vertical components of fluid velocities can be taken in a simple form, i.e., their dependence on the z coordinate in each layer is linear:

$$w_l = \frac{h_l + (-1)^l z}{h_l + (-1)^l \eta}\Big(\frac{\partial \eta}{\partial t} + (\boldsymbol{u}_{0i} + \boldsymbol{u}_i) \cdot \nabla \eta\Big). \tag{8}$$

For terms of the second order of smallness, the expressions are even simpler:

$$w_l = \Big(1 + (-1)^l \frac{z}{h_l}\Big)\Big(\frac{\partial \eta}{\partial t} + \boldsymbol{u}_{0i} \cdot \nabla \eta\Big) = \Big(1 + (-1)^l \frac{z}{h_l}\Big)\,\frac{D\eta}{D_0 t}. \tag{9}$$

In particular, relations (9) can be substituted not only into the integral terms of Eqs. (7), which take into account the inertia of each layer of the fluid, but also into the nonlinear terms $w_l\, \partial \boldsymbol{u}_l/\partial z$ of the equations of motion (5).

Dependences of Fluid Pressures and Velocities on Interface Disturbances. Substituting the formulas for velocities (1) and the normal components of velocities (9) into Eqs. (7), we determine the vertical pressure profiles:

$$\frac{p_l}{\rho_l} = \frac{p_{li}}{\rho_l} + g(\eta - z) + \int\limits_z^\eta \Big(1 + (-1)^l \frac{z}{h_l}\Big)(1 - A_l z^2 - B_l z)\boldsymbol{u}_{0i} \cdot \nabla \frac{D\eta}{D_0 t}\, dz$$

$$+ \int\limits_z^\eta \Big(1 + (-1)^l \frac{z}{h_l}\Big) \frac{\partial}{\partial t}\, \frac{D\eta}{D_0 t}\, dz = \frac{p_{li}}{\rho_l} + g(\eta - z) - \Big(z + \frac{z^2}{2}\, \frac{(-1)^l}{h_l}\Big) \frac{\partial}{\partial t}\, \frac{D\eta}{D_0 t}$$

$$- \Big[z + \frac{z^2}{2}\Big(\frac{(-1)^l}{h_l} - B_l\Big) - \frac{z^3}{3}\Big(A_l + B_l \frac{(-1)^l}{h_l}\Big) - \frac{z^4}{4}\, A_l\, \frac{(-1)^l}{h_l}\Big] \boldsymbol{u}_{0i} \cdot \nabla \frac{D\eta}{D_0 t}. \tag{10}$$

Then, applying the operator ∇ to Eq. (5) in a scalar manner and replacing $\nabla \cdot \boldsymbol{u}_l$ by $-\partial w_l/\partial z$ in the first three terms of Eq. (5) with the use of the continuity equation (4), we obtain

$$-\frac{\partial^2 w_l}{\partial t\, \partial z} - \boldsymbol{u}_{0l} \cdot \nabla \frac{\partial w_l}{\partial z} + \nabla w_l \cdot \frac{d\boldsymbol{u}_{0l}}{dz} + \nabla \cdot \Big(\boldsymbol{u}_l \cdot \nabla \boldsymbol{u}_l + w_l\, \frac{\partial \boldsymbol{u}_l}{\partial z}\Big)$$

$$+ \frac{1}{\rho_l}\, \nabla^2 p_l = \nu_l\, \frac{\partial^2}{\partial z^2}\, (\nabla \cdot \boldsymbol{u}_l). \tag{11}$$

Substituting the velocity profiles (1) and (8) into the second, third, and fourth terms of Eqs. (11) and dependences (10) into the fifth terms of Eqs. (11), we integrate Eqs. (11) with respect to the coordinate z from $-h_2$ to η for $l = 2$ and from η to h_1 for $l = 1$. As a result, we obtain

$$\frac{\partial w_i}{\partial t} + \frac{\partial \eta}{\partial t}\, \nabla \cdot \boldsymbol{u}_i + \frac{A_l}{3}\, h_l^2 \boldsymbol{u}_{0i} \cdot \nabla \Big[w_i \Big(1 + (-1)^l\, \frac{2\eta}{h_l}\Big)\Big] + \Big(\nabla \cdot \boldsymbol{u}_i - B_l\, \frac{D\eta}{D_0 t}\Big)\boldsymbol{u}_{0i} \cdot \nabla \eta$$

$$- [(-1)^l h_l + \eta]\Big(\frac{1}{\rho_l}\, \nabla^2 p_{li} + g \nabla^2 \eta\Big) - \frac{h_l^2}{3}\Big[\frac{\partial}{\partial t} + \frac{3}{4}\Big(1 + A_l\, \frac{h_l^2}{5}\Big)\boldsymbol{u}_{0i} \cdot \nabla\Big] \frac{D}{D_0 t}\, \nabla^2 \eta$$

$$- \int\limits_{-(-1)^l h_l}^{\eta} \nabla \cdot \Big((\boldsymbol{u}_l \cdot \nabla)\boldsymbol{u}_l + w_l\, \frac{\partial \boldsymbol{u}_l}{\partial z}\Big)\, dz = \frac{1}{\rho_l}\, (\nabla \cdot \boldsymbol{\tau}_{lz} - \nabla \cdot \boldsymbol{\tau}_{iz}) + \nu_l \nabla \eta \cdot \frac{\partial^2 \boldsymbol{u}_l}{\partial z^2}\Big|_{z=0}. \tag{12}$$

Here the shear stresses are $\boldsymbol{\tau}_{lz} = \nu_l \rho_l\, \partial \boldsymbol{u}_l/\partial z$ for $z = (-1)^l h_l$; corrections of the third order of smallness are omitted. To calculate the remaining integrals containing second-order terms only, it is sufficient to find the linear relation between the horizontal components of fluid velocities and the interface disturbance. Let us consider the boundary conditions and Eqs. (12) in the first approximation, i.e., with not only nonlinear, inertial, and capillary terms but also unsteady shear being neglected:

$$\Big(\frac{\partial}{\partial t} + A_l \,\frac{h_l^2}{3}\,\boldsymbol{u}_{0i}\cdot\nabla\Big)\frac{D\eta}{D_0 t} - (-1)^l h_l \Big(\frac{1}{\rho_l}\,\nabla^2 p_i + g\nabla^2\eta\Big) = 0. \tag{13}$$

From the condition of identity of these equations, which describe the same wave process, we determine the pressure Laplacian on the interface:

$$\nabla^2 p_{li} = -\rho_1\rho_2\Big(\frac{gH}{\chi}\,\nabla^2\eta + R_f\boldsymbol{u}_{0i}\cdot\nabla\frac{D\eta}{D_0 t}\Big). \tag{14}$$

Here $\chi = \rho_1 h_2 + \rho_2 h_1$ and $R_f = (\nu_1\rho_1 h_2 + \nu_2\rho_2 h_1)(\nu_2\rho_2 h_1^2 - \nu_1\rho_1 h_2^2)/(3\nu_1\nu_2\rho_1\rho_2 h_1 h_2\chi H)$. If the main disturbances propagate only in a certain direction (the vector of the phase velocity of linear waves $\boldsymbol{c}$ is almost parallel to the vector $\nabla\eta$), we can replace the operators

$$\nabla\eta = -\frac{\boldsymbol{c}}{c^2}\,\frac{\partial\eta}{\partial t}, \qquad \nabla^2 = \frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}.$$

Then Eq. (14) can be written as the dependence

$$\frac{\partial^2 p_{li}}{\partial t^2} = -\rho_1\rho_2\Big[\frac{gH}{\chi} - R_f\boldsymbol{u}_{0i}\cdot\boldsymbol{c}\Big(1 - \frac{\boldsymbol{u}_{0i}\cdot\boldsymbol{c}}{c^2}\Big)\Big]\frac{\partial^2\eta}{\partial t^2}.$$

Integrating this equation once with respect to time, we obtain

$$\frac{\partial p_{li}}{\partial t} = -\rho_1\rho_2\Big(\frac{gH}{\chi} - R_f u_{0c}^2\Big)\frac{\partial\eta}{\partial t}, \qquad u_{0c}^2 = \boldsymbol{u}_{0i}\cdot\boldsymbol{c}\Big(1 - \frac{\boldsymbol{u}_{0i}\cdot\boldsymbol{c}}{c^2}\Big).$$

Here the constant of integration is assumed to be zero, because $\partial p_i/\partial t = 0$ in the absence of disturbances, as it follows from the previously made assumption. As a result, we have

$$\nabla p_{li} = -\rho_1\rho_2\Big(\frac{gH}{\chi} - R_f u_{0c}^2\Big)\nabla\eta. \tag{15}$$

We integrate the linearized and dissipation-free horizontal components of the equations of motion (5) with respect to the vertical coordinate z from $-h_2$ to η for $l = 2$ and from η to h_1 for $l = 1$. We substitute the velocity profiles (1) into the second and third terms of these equations, the velocity profiles (9) into the third terms of these equations, and the hydrostatic relations $p_l = p_{li} + \rho_l g(\eta - z)$ and expressions (15) into the last terms of these equations. As a result, we obtain the equations of motion in the horizontal plane

$$\Big[1 - \frac{S_c}{2}\Big(1 + \frac{A_l}{3}\,h_l^2\Big)\Big]\frac{\partial\boldsymbol{u}_l}{\partial t} + \Big[\frac{\boldsymbol{c}}{c^2}\,g_l + (-1)^l\,\frac{\boldsymbol{u}_{0i}}{2h_l}\,\frac{c_f^2}{c^2}\Big(1 - \frac{A_l}{3}\,h_l^2\Big)\Big]\frac{\partial\eta}{\partial t} = 0, \tag{16}$$

where $g_l = \rho_1\rho_2(gH/\chi - R_f u_{0c}^2)/\rho_l - g$, $S_c = \boldsymbol{u}_{0i}\cdot\boldsymbol{c}/c^2$, and $c_f^2 = c^2(1 - S_c)$. Note that $\boldsymbol{u}_l$ is independent of the vertical coordinate in the approximation considered. Integrating Eqs. (16) with respect to time, we find the relations between the horizontal velocities of the fluids and the interface disturbance:

$$\boldsymbol{u}_l = -\frac{\boldsymbol{c}g_l h_l + (-1)^l\boldsymbol{u}_{0i}c_f^2(1 - A_l h_l^2/3)/2}{h_l c^2[1 - S_c(1 + A_l h_l^2/3)/2]}\,\eta = \frac{\boldsymbol{c}_l}{h_l}\,\eta. \tag{17}$$

As $\boldsymbol{u}_l = 0$ in the absence of disturbances under the assumptions made, the constant of integration is again set to zero.

Determining the Phase Velocity and Shear Stresses on the Boundaries. To determine the phase velocity $\boldsymbol{c}$ as a function of the magnitude and direction of the steady flow, we substitute $\nabla^2 p_{li}$ from Eq. (14) into system (13). We obtain the equation for the interface disturbance and seek for its solution in the form of a linear monoharmonic wave propagating at an angle φ to the steady flow direction vector $\boldsymbol{u}_{0i}$. Thus, the absolute value of the phase velocity is described by the formula

$$c = \Big||\boldsymbol{u}_{0i}|\cos\varphi(1 + S_f)/2 \pm \sqrt{c_0^2 + \boldsymbol{u}_{0i}^2[\cos\varphi(1 - S_f)/2]^2}\,\Big|,$$

where $S_f = (\nu_1 h_2 + \nu_2 h_1)(\nu_1\rho_1 h_2 + \nu_2\rho_2 h_1)/(3\nu_1\nu_2\chi H)$ and $c_0^2 = gh_1h_2(\rho_2 - \rho_1)/\chi$. As it was expected, the cocurrent flow increases the phase velocity of disturbances, while the counterflow decreases the phase velocity.

To find the shear stresses $\boldsymbol{\tau}_{lz}$ on all boundaries, we use the equations of motion (5) with nonlinear and inertial terms being neglected:

$$\frac{\partial \boldsymbol{u}_l}{\partial t} + \boldsymbol{u}_{0l}\cdot\nabla\boldsymbol{u}_l + w_l\,\frac{d\boldsymbol{u}_{0l}}{dz} + g\nabla\eta + \frac{1}{\rho_l}\,\nabla p_{li} = \nu_l\,\frac{\partial^2\boldsymbol{u}_l}{\partial z^2}. \tag{18}$$

Using formulas (1) for $\boldsymbol{u}_{0l}$, formulas (9) for w_l, and relations (15) for ∇p_{li}, and replacing $\boldsymbol{u}_{0l}\cdot\nabla\boldsymbol{u}_l$ by $-(\boldsymbol{u}_{0l}\cdot\boldsymbol{c}/c^2)(\partial\boldsymbol{u}_l/\partial t)$, we write the equations of motion (18) in the form

$$\Big(1-\frac{\boldsymbol{u}_{0i}\cdot\boldsymbol{c}}{c^2}\,(1-A_lz^2-B_lz)\Big)\frac{\partial\boldsymbol{u}_l}{\partial t} - \boldsymbol{u}_{0i}(2A_lz+B_l)\Big(1+(-1)^l\frac{z}{h_l}\Big)\frac{D\eta}{D_0t}$$

$$+\Big((-1)^l\,\frac{c_0^2}{h_l} + \rho_1\rho_2\,\frac{R_f}{\rho_l}\,u_{0c}^2\Big)\nabla\eta = \nu_l\,\frac{\partial^2\boldsymbol{u}_l}{\partial z^2}. \tag{19}$$

Replacing $\partial\eta/\partial t$ in these equations by $-\boldsymbol{c}\cdot\nabla\eta$ and dividing all terms by ν_l, we obtain

$$\frac{\partial^2\boldsymbol{u}_l}{\partial z^2} - \frac{1}{\nu_l}\Big(1-\frac{\boldsymbol{u}_{0i}\cdot\boldsymbol{c}}{c^2}\,(1-A_lz^2-B_lz)\Big)\frac{\partial\boldsymbol{u}_l}{\partial t}$$

$$= -\frac{g_l}{\nu_l}\,\nabla\eta + \frac{\boldsymbol{u}_{0i}}{\nu_l}\,(2A_lz+B_l)\Big(1+(-1)^l\,\frac{z}{h_l}\Big)[(\boldsymbol{c}-\boldsymbol{u}_{0i})\cdot\nabla\eta]. \tag{20}$$

The solutions of Eqs. (20) are sought by the method of separation of variables:

$$\boldsymbol{u}_l(t,x,y,z) = \boldsymbol{u}_l'(t,z)f_l(x,y).$$

First we consider the boundary layers near the lid and bottom. In this case, $z\approx-(-1)^lh_l$ and the right sides of Eqs. (20) are substantially simplified (become independent of z). Hence, applying the Laplace transform in time to Eqs. (20), we obtain

$$\frac{\partial^2\boldsymbol{V}_l(s,z)}{\partial z^2} - \frac{s}{\nu_l}\,\boldsymbol{V}_l(s,z) = \frac{\boldsymbol{G}_{\nabla l}(s,x,y)}{\nu_lf_l(x,y)} - \frac{\boldsymbol{u}_{l0}'}{\nu_l} \equiv \boldsymbol{P}_{\nabla l}(s). \tag{21}$$

Here $\boldsymbol{V}_l(s,z)$ and $\boldsymbol{G}_{\nabla l}(s,x,y)/\nu_l$ are the images of $\boldsymbol{u}_l'(t,z)$ and the right sides of Eqs. (2) for $z=-(-1)^lh_l$. The right sides of Eqs. (21) are functions of the variable s only, because their left sides are independent of the coordinates x and y, and the fluid velocities in each layer are independence of the vertical coordinate z at the initial time $t=0$. Without decreasing accuracy, the assumption about a small thickness of the boundary layers allows us to pose a condition of the absence of shear stresses at large distances from the surfaces considered: as $z=-\infty$ for the lid, as $z=+\infty$ for the bottom, and as $z=-\infty$ and $z=+\infty$ for the interface (the boundary layers are almost "infinitely" deeply immersed into the fluid).

After the replacement $\boldsymbol{V}_l'=\boldsymbol{V}_l+\boldsymbol{P}_{\nabla l}\nu_l/s$, we write Eqs. (21) in the form of linear homogeneous equations. Then, we can readily find the solutions satisfying the boundary conditions $\boldsymbol{V}_l=0$ for $z=-(-1)^lh_l$ and $\partial\boldsymbol{V}_l/\partial z=0$ for $z=(-1)^l\infty$:

$$\boldsymbol{V}_l(s,z) = \boldsymbol{P}_{\nabla l}(s)\,\frac{\nu_l}{s}\Big[\exp\Big(-\sqrt{\frac{s}{\nu_l}}\,[(-1)^lz+h_l]\Big)-1\Big].$$

From here, we find the derivative at the lid and at the bottom:

$$\frac{\partial\boldsymbol{V}_l}{\partial z}\Big|_{z=-(-1)^lh_l} = -(-1)^l\boldsymbol{P}_{\nabla l}(s)\sqrt{\frac{\nu_l}{s}}.$$

Applying the inverse Laplace transform to these formulas, we obtain a convolution-type expression for viscous shear stresses on the lid and bottom in the space of originals:

$$\boldsymbol{\tau}_{lz} = \nu_l\rho_l\,\frac{\partial\boldsymbol{u}_l}{\partial z}\Big|_{z=-(-1)^lh_l} = (-1)^l\sqrt{\nu_l}\rho_l\Big(\frac{g_l}{\sqrt{\pi}}\int\limits_0^t\frac{\nabla\eta\,dt'}{\sqrt{t-t'}} + \frac{1}{\sqrt{\pi t}}\,\boldsymbol{u}_{l0}(x,y)\Big). \tag{22}$$

Note that $t>0$ in Eqs. (22), and the terms containing the variables $\boldsymbol{u}_{l,0}(x,y)$ and $\eta_0(x,y)$ exert some effect only in the region disturbed at the initial time. The influence of these terms on the remaining space is insignificant.

We consider the boundary layers near the interface [$z \approx 0$ in the right sides of Eqs. (19), which also cease to depend on the z coordinate]. Moreover, replacing $\nabla\eta$ by $-(\boldsymbol{c}/c^2)\,\partial\eta/\partial t$, we obtain the equations of horizontal motion

$$\frac{\partial^2 \boldsymbol{u}_l}{\partial z^2} - \frac{1}{\nu_l}\,\frac{c_f^2}{c^2}\,\frac{\partial \boldsymbol{u}_l}{\partial t} = \Big(\frac{\boldsymbol{c}}{c^2}\,\frac{g_l}{\nu_l} - \boldsymbol{u}_{0i}\,\frac{B_l}{\nu_l}\,\frac{c_f^2}{c^2}\Big)\frac{\partial\eta}{\partial t}.$$

Using the Laplace transform in time, we write these equations in the form

$$\frac{\partial^2 \boldsymbol{V}_l}{\partial z^2} - \boldsymbol{V}_l\,\frac{s}{\nu_l}\,\frac{c_f^2}{c^2} = \frac{\boldsymbol{G}_{tl}}{\nu_l f_l} - \frac{\boldsymbol{u}_{l0}}{\nu_l}\,\frac{c_f^2}{c^2} \equiv \boldsymbol{P}_{tl}(s). \tag{23}$$

Based on the joint solution of Eqs. (23) with the boundary conditions $\partial\boldsymbol{V}_l/\partial z = 0$ for $z = -(-1)^l\infty$ (because these boundary layers are also almost "infinitely" deeply immersed into the fluid layers), $\boldsymbol{V}_1 = \boldsymbol{V}_2$ and $\nu_1\rho_1\,\partial\boldsymbol{V}_1/\partial z = \nu_2\rho_2\,\partial\boldsymbol{V}_2/\partial z$ for $z = 0$ (continuity of velocities and shear stresses), we find the profiles for the images:

$$\boldsymbol{V}_l(s,z) = \frac{(-1)^l}{s_c}\,\frac{\psi}{\psi_l}\,[\nu_2\boldsymbol{P}_{t2}(s) - \nu_1\boldsymbol{P}_{t1}(s)]\exp\Big((-1)^l\sqrt{\frac{s_c}{\nu_l}}\,z\Big) - \frac{\nu_l}{s_c}\,\boldsymbol{P}_{tl}(s).$$

Here $s_c = sc_f^2/c^2$, $\psi_l = \sqrt{\nu_l}\rho_l$, and $\psi = \psi_1\psi_2/(\psi_1+\psi_2)$. With repeated application of the inverse Laplace transform, replacement for the gradient operator, and simple equalities

$$\int_0^t \nabla\eta(t',x,y)\,dt' = -\frac{\boldsymbol{c}}{c^2}\int_0^t \frac{\partial\eta(t',x,y)}{\partial t'}\,dt' = \frac{\boldsymbol{c}}{c^2}\,[\eta_0(x,y) - \eta(t,x,y)];$$

these formulas yield the expressions for the horizontal component of velocity of the fluid particles and viscous shear stresses on the interface:

$$\boldsymbol{u}_i = [\eta(t,x,y) - \eta_0(x,y)]\boldsymbol{f}_s + \frac{\psi_1\boldsymbol{u}_{1,0}(x,y) + \psi_2\boldsymbol{u}_{2,0}(x,y)}{\psi_1+\psi_2},$$

$$\boldsymbol{f}_s = \boldsymbol{u}_{0i}\,\frac{\psi_1 B_1 + \psi_2 B_2}{\psi_1+\psi_2} + \boldsymbol{c}\Big(\frac{c_0^2}{c_f^2}\,\frac{\psi_2 h_1 - \psi_1 h_2}{h_1 h_2(\psi_1+\psi_2)} + \boldsymbol{u}_{0i}\cdot\boldsymbol{c}\,\frac{R_f}{c^2}\,\frac{\rho_1\psi_2 + \rho_2\psi_1}{\psi_1+\psi_2}\Big); \tag{24}$$

$$\boldsymbol{\tau}_{iz} = \nu_l\rho_l\,\frac{\partial\boldsymbol{u}_l}{\partial z}\Big|_{z=0} = \frac{\psi}{\sqrt{\pi t}}\,\frac{c_f}{c}\,[\boldsymbol{u}_{1,0}(x,y) - \boldsymbol{u}_{2,0}(x,y)]$$

$$+\frac{\psi}{\sqrt{\pi}}\,\frac{c_f}{c}\Big[\boldsymbol{u}_{0i}(B_1 - B_2) - \boldsymbol{c}\Big(\frac{c_0^2 H}{c_f^2 h_1 h_2} - (\rho_2-\rho_1)\boldsymbol{u}_{0i}\cdot\boldsymbol{c}\,\frac{R_f}{c^2}\Big)\Big]\int_0^t \frac{\partial\eta(t',x,y)}{\partial t'}\,\frac{dt'}{\sqrt{t-t'}}. \tag{25}$$

Note that $t > 0$ in Eqs. (24) and (25), and the terms containing the variables $\boldsymbol{u}_{l,0}$ and η_0 affect only the region disturbed at the initial time.

Finally, from Eqs. (18) for $z = 0$, we find the expressions for the second derivatives of unsteady velocities of the fluids over the vertical coordinate near the interface of the layers, which enter the right sides of Eqs. (12):

$$\nu_l\,\frac{\partial^2\boldsymbol{u}_l}{\partial z^2}\Big|_{z=0} = \frac{\partial\boldsymbol{u}_i}{\partial t} + \boldsymbol{u}_{0i}\cdot\nabla\boldsymbol{u}_i - \boldsymbol{u}_{0i}B_l\,\frac{D\eta}{D_0 t} - g_l\nabla\eta. \tag{26}$$

Thus, we have all relations necessary to derive the equation for the disturbance $\eta(t,x,y)$.

Evolutionary Equation for Waves on the Interface. Substituting dependences (15), (17), and (26) into the second-order terms of Eqs. (12), we obtain

$$\frac{\partial^2\eta}{\partial t^2} + \boldsymbol{u}_{0i}\cdot\nabla\frac{\partial\eta}{\partial t} - \boldsymbol{u}_{0i}\cdot\nabla(\boldsymbol{u}_i\cdot\nabla\eta) + \frac{A_l}{3}\,h_l^2\boldsymbol{u}_{0i}\cdot\nabla\Big[\Big(1 + (-1)^l\,\frac{2\eta}{h_l}\Big)\frac{D\eta}{D_0 t} + \boldsymbol{u}_i\cdot\nabla\eta\Big]$$

$$-\frac{1}{2h_l}\Big((-1)^l(\boldsymbol{c}_l\cdot\nabla)^2\eta^2 - \boldsymbol{c}_l\cdot\nabla\frac{D\eta^2}{D_0 t}\Big) - (-1)^l\,\frac{h_l}{\rho_l}\,\nabla^2 p_{li} - (-1)^l g h_l\nabla^2\eta + \frac{g_l}{2}\,\nabla^2\eta^2$$

$$-\frac{h_l^2}{3}\Big[\frac{\partial}{\partial t} + \frac{3}{4}\Big(1 + A_l\,\frac{h_l^2}{5}\Big)\boldsymbol{u}_{0i}\cdot\nabla\Big]\frac{D}{D_0 t}\,\nabla^2\eta = \frac{1}{\rho_l}\,\nabla\cdot(\boldsymbol{\tau}_{lz} - \boldsymbol{\tau}_{iz}). \tag{27}$$

In (27), a series of terms is grouped, which are integrated by parts:

$$\int\limits_{-(-1)^l h_l}^{0} w_l \frac{\partial \boldsymbol{u}_l}{\partial z}\,dz = \boldsymbol{u}_i \frac{D\eta}{D_0 t} - \int\limits_{-(-1)^l h_l}^{0} \boldsymbol{u}_l \frac{\partial w_l}{\partial z} dz = \boldsymbol{u}_i \frac{D\eta}{D_0 t} - \frac{\boldsymbol{c}_l}{2h_l}\frac{D\eta^2}{D_0 t}.$$

To reduce system (27) to one equation (eliminating pressure on the interface from the remaining linear terms), we multiply Eq. (27) by h_2/ρ_2 for $l = 1$ and by h_1/ρ_1 for $l = 2$ and then sum up the resultant expressions to obtain

$$\frac{\partial^2\eta}{\partial t^2} - c_0^2\nabla^2\eta + (1+S_f)\boldsymbol{u}_{0i}\cdot\nabla\frac{\partial\eta}{\partial t} + S_f(\boldsymbol{u}_{0i}\cdot\nabla)^2\eta - (1-S_f)\boldsymbol{u}_{0i}\cdot\nabla(\boldsymbol{u}_i\cdot\nabla\eta)$$
$$- \Big[H_\rho^2\Big(\frac{1}{3}\frac{\partial^2}{\partial t^2} + \frac{7}{12}\boldsymbol{u}_{0i}\cdot\nabla\frac{\partial}{\partial t} + \frac{1}{4}(\boldsymbol{u}_{0i}\cdot\nabla)^2\Big) + S_A\boldsymbol{u}_{0i}\cdot\nabla\frac{D}{D_0 t} - \sigma\frac{h_1 h_2}{\chi}\nabla^2\Big]\nabla^2\eta$$
$$+ \frac{\rho_1 h_2}{2h_1\chi}\Big(\frac{D}{D_0 t} + \boldsymbol{c}_1\cdot\nabla\Big)\boldsymbol{c}_1\cdot\nabla\eta^2 + \frac{\rho_2 h_1}{2h_2\chi}\Big(\frac{D}{D_0 t} - \boldsymbol{c}_2\cdot\nabla\Big)\boldsymbol{c}_2\cdot\nabla\eta^2$$
$$- C_{N\Delta}\nabla^2\eta^2 - R_\nu\boldsymbol{u}_{0i}\cdot\nabla\frac{D\eta^2}{D_0 t} = \frac{1}{\chi}\nabla\cdot(h_1\boldsymbol{\tau}_{2z} + h_2\boldsymbol{\tau}_{1z} - H\boldsymbol{\tau}_{iz}), \qquad (28)$$

where $S_A = h_1h_2(\rho_1h_1^3A_1 + \rho_2h_2^3A_2)/(20\chi)$, $C_{N\Delta} = [c_0^2(\rho_2h_1/h_2 - \rho_1h_2/h_1) + u_{0c}^2\rho_1\rho_2R_fH]/(2\chi)$, $R_\nu = (\nu_2 - \nu_1)(\nu_1\rho_1h_2 + \nu_2\rho_2h_1)/(3\nu_1\nu_2\chi H)$, and $H_\rho^2 = h_1h_2(\rho_1h_1 + \rho_2h_2)/\chi$.

With allowance for the assumptions made on the long-wave character of low-amplitude disturbances and on unidirectional propagation of these disturbances, the following replacements can be performed in second-order terms: $\partial\eta/\partial t$ by $-\boldsymbol{c}\cdot\nabla\eta$ in nonlinear terms, ∇ by $-(\boldsymbol{c}/c^2)\partial/\partial t$ in dispersion terms, and ∇^2 by $(1/c^2)\,\partial^2/\partial t^2$ in inertial and capillary terms. Then, Eq. (28) can be written as

$$\frac{\partial^2\eta}{\partial t^2} - c_0^2\nabla^2\eta + (1+S_f)\boldsymbol{u}_{0i}\cdot\nabla\frac{\partial\eta}{\partial t} + S_f(\boldsymbol{u}_{0i}\cdot\nabla)^2\eta - (1-S_f)\boldsymbol{u}_{0i}\cdot\nabla(\boldsymbol{u}_i\cdot\nabla\eta)$$
$$- C_d\frac{\partial^2}{\partial t^2}\nabla^2\eta + \frac{\rho_1h_2}{2h_1\chi}(\boldsymbol{u}_{0i} - \boldsymbol{c} + \boldsymbol{c}_1)\cdot\nabla(\boldsymbol{c}_1\cdot\nabla\eta^2) + \frac{\rho_2h_1}{2h_2\chi}(\boldsymbol{u}_{0i} - \boldsymbol{c} - \boldsymbol{c}_2)\cdot\nabla(\boldsymbol{c}_2\cdot\nabla\eta^2)$$
$$- C_{N\Delta}\nabla^2\eta^2 - R_\nu\boldsymbol{u}_{0i}\cdot\nabla[(\boldsymbol{u}_{0i} - \boldsymbol{c})\cdot\nabla\eta^2] = \frac{1}{\chi}\nabla\cdot(h_1\boldsymbol{\tau}_{2z} + h_2\boldsymbol{\tau}_{1z} - H\boldsymbol{\tau}_{iz}), \qquad (29)$$

where $C_d = H_\rho^2(1/3 - 7S_c/12 + S_c^2/4) + S_A(S_c^2 - S_c) - \sigma h_1h_2/(c^2\chi)$. Substituting the expressions for shear stresses on all boundaries of system (22), (25) and expressions for velocities near the interface (24) into Eq. (29), we obtain only three unknowns in the resultant expression: η and $\boldsymbol{u}_{l0}$ ($l = 1, 2$) [the initial disturbance $\eta_0(x, y)$ is assumed to be known], and $\boldsymbol{u}_{l0}$ enter only terms of the second order of smallness. Hence, $\boldsymbol{u}_{l0}$ can be replaced by $\boldsymbol{c}_l\eta_0/h_l$ in Eq. (29), which yields the basic evolutionary equation for the disturbance:

$$\frac{\partial^2\eta}{\partial t^2} + (1+S_f)\boldsymbol{u}_{0i}\cdot\nabla\frac{\partial\eta}{\partial t} - c_0^2\nabla^2\eta + S_f(\boldsymbol{u}_{0i}\cdot\nabla)^2\eta - C_d\frac{\partial^2}{\partial t^2}\nabla^2\eta - C_{Nxx}\frac{\partial^2\eta^2}{\partial x^2}$$
$$- C_{Nxy}\frac{\partial^2\eta^2}{\partial x\partial y} - C_{Nyy}\frac{\partial^2\eta^2}{\partial y^2} + \int\limits_0^t\Big(C_{Bxx}\frac{\partial^2\eta}{\partial x^2} + C_{Bxy}\frac{\partial^2\eta}{\partial x\,\partial y} + C_{Byy}\frac{\partial^2\eta}{\partial y^2}\Big)\frac{dt'}{\sqrt{t-t'}}$$
$$= (\boldsymbol{f}_{N0}\cdot\nabla)(\eta_0\boldsymbol{u}_{0i}\cdot\nabla\eta) + \frac{1}{\sqrt{\pi t}}\boldsymbol{f}_{L0}\cdot\nabla\eta_0. \qquad (30)$$

The coefficients in this equation are determined only by geometric (h_1, h_2, and φ) and physical ($g, \rho_1, \rho_2, \nu_1, \nu_2, \boldsymbol{u}_{0i}$, and σ) parameters of the problem considered:

$$C_{Nxx} = C_{N\Delta} - R_\nu u_{0i}(c_x - u_{0i}) + \frac{R_S}{2}u_{0i}f_{sx} + \frac{k_{2x}d_{2x} - k_{1x}d_{1x}}{2h_1h_2\chi},$$

$$C_{Nxy} = \frac{R_S}{2}(u_{0i}f_{sy} + v_{0i}f_{sx}) - R_\nu[u_{0i}(c_y - v_{0i}) + v_{0i}(c_x - u_{0i})] + \frac{k_{2x}d_{2y} + k_{2y}d_{2x} - k_{1x}d_{1y} - k_{1y}d_{1x}}{2h_1h_2\chi}, \qquad R_S = 1 - S_f,$$

$$C_{Nyy} = C_{N\Delta} - R_\nu v_{0i}(c_y - v_{0i}) + \frac{R_S}{2}\, v_{0i} f_{sy} + \frac{k_{2y}d_{2y} - k_{1y}d_{1y}}{2h_1h_2\chi},$$

$$\boldsymbol{k}_1 = \boldsymbol{c}_1\rho_1 h_2^2, \qquad \boldsymbol{k}_2 = \boldsymbol{c}_2\rho_2 h_1^2, \qquad \boldsymbol{d}_1 = \boldsymbol{c}_1 - \boldsymbol{c} + \boldsymbol{u}_{0i}, \qquad \boldsymbol{d}_2 = \boldsymbol{c}_2 + \boldsymbol{c} - \boldsymbol{u}_{0i},$$

$$C_{Bxx} = C_{B\Delta} + \frac{\psi_H}{\sqrt{\pi}\chi}\, c_x f_{Bx}, \qquad C_{B\Delta} = \frac{\psi_1 g_1 h_2 - \psi_2 g_2 h_1}{\sqrt{\pi}\chi}, \qquad \psi_H = \psi H\, \frac{c_f}{c},$$

$$C_{Bxy} = \frac{\psi_H}{\sqrt{\pi}\chi}\,(c_x f_{By} + c_y f_{Bx}), \qquad C_{Byy} = C_{B\Delta} + \frac{\psi_H}{\sqrt{\pi}\chi}\, c_y f_{By},$$

$$\boldsymbol{f}_B = \frac{\boldsymbol{c} c_0^2 H}{c_f^2 h_1 h_2} + \boldsymbol{u}_{0i}(B_2 - B_1) - R_f(\rho_2 - \rho_1)\,\frac{\boldsymbol{c}}{c^2}\,\boldsymbol{u}_{0i}\cdot\boldsymbol{c},$$

$$\boldsymbol{f}_{L0} = \frac{\psi_H(\boldsymbol{c}_2 h_1 - \boldsymbol{c}_1 h_2) - \psi_1\boldsymbol{c}_1 h_2^2 + \psi_2\boldsymbol{c}_2 h_1^2}{h_1h_2\chi}, \qquad \boldsymbol{f}_{N0} = R_S\Big(\frac{\psi_1\boldsymbol{c}_1 h_2 + \psi_2\boldsymbol{c}_2 h_1}{h_1h_2(\psi_1+\psi_2)} - \boldsymbol{f}_s\Big).$$

The model equation (30) takes into account not only the long-wave contributions of inertia of the fluid layers and surface tension, weak nonlinearity of disturbances and unsteady shear stresses on all boundaries of this system, but also the steady flow of the viscous fluid in the horizontal channel. Note that the evolutionary equation (30) can also be used to describe transformation of waves propagating in an arbitrary horizontal direction (at an arbitrary angle to the flow velocity vector), while the terms in the right side of this equation differ from zero only in the region of the initial disturbance.

Numerical Solutions of Problems on Transformation of Various Waves. Some results of calculations based on an evolutionary equation similar to Eq. (30) are given in [6]. These calculations are performed with the use of an implicit three-layer finite-difference scheme. Our model equations differ not only by the form of coefficients, but also by the fact that Eq. (30) contains terms with mixed second derivatives. To eliminate terms containing one derivative with respect to time and one derivative with respect to the horizontal coordinate, we pass to a reference system moving with a velocity $u_f = u_{0i}(1+S_f)/2$ in the direction with increasing x coordinate and with a velocity $v_f = v_{0i}(1+S_f)/2$ in the direction with increasing y coordinate. Then Eq. (2) can be written in the form

$$\frac{\partial^2\eta}{\partial t_f^2} - c_0^2\nabla_f^2\eta + S_f(\boldsymbol{u}_{0i}\cdot\nabla_f)^2\eta - (\boldsymbol{u}_f\cdot\nabla_f)^2\eta - \frac{C_d}{R_{uv}^2}\,\frac{\partial^2}{\partial t_f^2}\Big(\frac{\partial^2\eta}{\partial x_f^2} + \frac{\partial^2\eta}{\partial y_f^2}\Big) - C_{Nxx}\,\frac{\partial^2\eta^2}{\partial x_f^2}$$

$$- C_{Nxy}\,\frac{\partial^2\eta^2}{\partial x_f\partial y_f} - C_{Nyy}\,\frac{\partial^2\eta^2}{\partial y_f^2} - \int\limits_0^{t_f}\Big(C_{Bxx}\,\frac{\partial^2\eta}{\partial x_f^2} + C_{Bxy}\,\frac{\partial^2\eta}{\partial x_f\partial y_f} + C_{Byy}\,\frac{\partial^2\eta}{\partial y_f^2}\Big)\frac{dt'}{\sqrt{t_f - t'}}$$

$$= \frac{1}{\sqrt{\pi t}}\,\boldsymbol{f}_{L0}\cdot\nabla_f\eta_0 + (\boldsymbol{f}_{N0}\cdot\nabla_f)(\eta_0\boldsymbol{u}_{0i}\cdot\nabla_f\eta). \tag{31}$$

Here $t_f = t$, $\nabla_f = (\partial/\partial x_f, \partial/\partial y_f)$, $R_{uv} = 1 - \boldsymbol{c}\cdot\boldsymbol{u}_f/c^2$, $x_f = x - u_f t$, and $y_f = y - v_f t$. Now Eq. (31) differs from the equation derived in [6] by the fact that Eq. (31) ignores the bottom slope, while the equation [6] does not contain terms similar to terms with mixed second derivatives with respect to the horizontal coordinates and the last term in the right side of this equation (schemes for these derivatives, based on central differences, are commonly known). Figures 2–4 show the evolution of solitary disturbances calculated with steps of 2 cm in the x direction, 10 cm in the y direction, and 0.5 sec in time.

First we consider problems where the initial disturbances (in the first and second time layers) are plane (independent of the y coordinate) solitary waves

$$\eta = \eta_s\,\mathrm{secosh}^2(\xi/L), \tag{32}$$

similar to a soliton (dissipation-free) solution of the one-dimensional equation (29):

$$L = L_s = U\sqrt{\frac{6C_d}{\eta_a C_{Nxx}}}, \qquad U = \frac{u_{0i}(1+S_f)}{2} + \sqrt{\frac{u_{0i}^2(1-S_f)^2}{4} + c_0^2 + \frac{2}{3}\,\eta_a C_{Nxx}} \qquad (\xi = x - Ut).$$

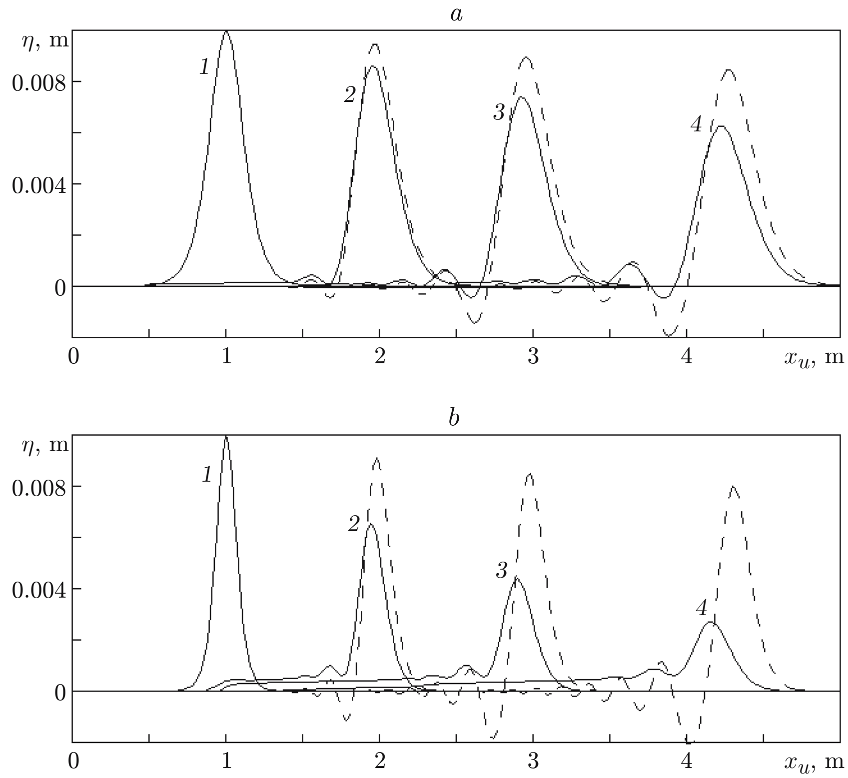

Fig. 2. Profiles of moderately long disturbances at different times t ($h_1 = 5$ cm, $h_1/h_2 = 5/4$, $\rho_1 = 1$ g/cm^3, $\rho_1/\rho_2 = 0.98$, $\nu_1 = 1$ mm^2/sec, $\nu_1/\nu_2 = 0.23$, and $\sigma = 45$ mN/m): $u_0^* = -0.5$ (a) and 0.5 (b); the solid and dashed curves are the calculations with allowance for unsteady shear stresses and without allowance for dissipation, respectively; $t = 0$ (1), 15 (2), 30 (3), and 40 sec (4).

Hence, it is no problem to define the partial derivative with respect to time at the initial moment. Note that the components of velocity of the horizontal flow v_{0l} produce some effect only if the derivatives with respect to the y coordinate differ from zero.

If the initial disturbances are moderately long [dependence (32) with $L = L_s/2$], their leading fronts become less steep, and slowly decaying oscillations arise behind the fronts (see Fig. 2). As in the case without a steady flow (see [7]), not only the disturbance amplitude decreases, but also dissipative tails appear. Figure 2 shows the effect of unsteady shear stresses on the boundaries, because the value of the main dissipative coefficient C_{Bxx} at $u_0^* = -0.5$ is approximately half its value at $u_0^* = 0.5$ (see [5]). For greater viscous losses, the oscillations behind the main wave practically disappear, and disturbances do not change their sign.

If the initial wave is sufficiently long [dependence (32) with $L = 2L_s$], then, in the absence of dissipation, the initial disturbance first takes the form of a triangular disturbance with a steep leading front and an extended rear front, and later it transforms to a chain of solitary waves with decreasing amplitude (see Fig. 3). Allowance for unsteady shear stresses does not allow the primary disturbance to grow and decelerates the formation of the chain of solitons.

Let us consider the evolution of a weakly nonlinear wave, which is solitary in space. Figure 4 shows the results calculated for the initial disturbance

$$\eta = \eta_s \operatorname{secosh}^2(\xi_x/L_s) \operatorname{secosh}^2(0.25\xi_y/L_s),$$

where $\xi_x = x - tU\cos\alpha$ and $\xi_y = y - tU\sin\alpha$ (α is the angle between the wave-propagation direction and the Ox axis). The disturbance is seen to transform to a horse-shoe wave. Owing to the presence of a steady flow in the transverse direction, the disturbance decays more intensely. A change in the initial direction of wave motion mainly affects the asymmetry of the "horse-shoe arcs."

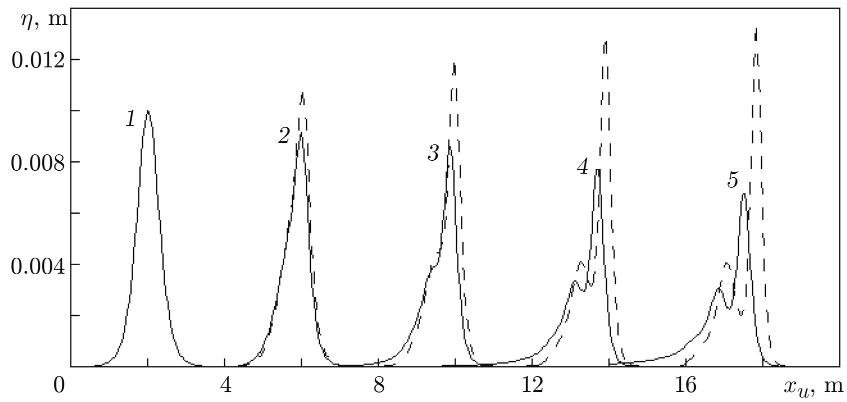

Fig. 3. Profiles of sufficiently long waves at different times t ($h_1 = 6$ cm, $h_1/h_2 = 2$, $\rho_1 = 1$ g/cm^3, $\rho_1/\rho_2 = 0.98$, $\nu_1 = 1$ mm^2/sec, $\nu_1/\nu_2 = 0.23$, $\sigma = 45$ mN/m, and $u_0^* = -0.5$): the solid and dashed curves are the calculations with allowance for unsteady shear stresses and without allowance for dissipation, respectively; $t = 0$ (1), 60 (2), 120 (3), 180 (4), and 240 sec (5).

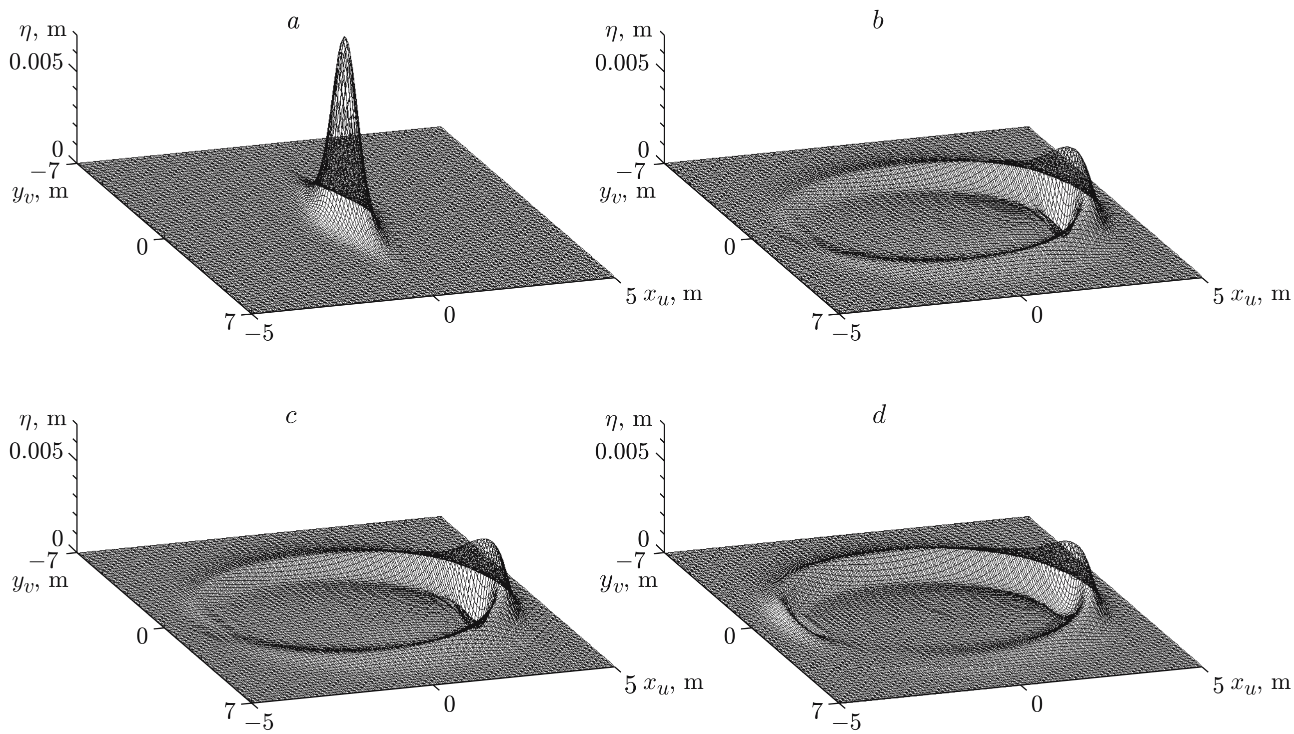

Fig. 4. Initial three-dimensional solitary disturbance (a) and wave shapes at $t = 60$ sec (b–d): (b) $\alpha = 0$, $u_0^* = -0.5$, and $v_0^* = 0$; (c) $\alpha = 0$, $u_0^* = -0.5$, and $v_0^* = -0.5$; (d) $\alpha = -\pi/4$, $u_0^* = -0.5$, and $v_0^* = -0.5$ (the remaining parameters the same as in Fig. 2).

Similar disturbances were previously observed on the surface of downward flowing liquid films (see, e.g., [8–10]), but the physics of the process is essentially different in these situations. The force returning the disturbed horizontal boundary to the equilibrium position is the force of gravity, while the dispersion is predominantly caused by the inertia of the layers and surface tension. For waves on the free boundary of the liquid film, the balance of the force of gravity and shear stresses on the vertical solid wall generates a steady flow and determines the phase velocity of disturbances. In addition, the waves on such a surface can move only in the downstream direction, while the disturbances of the horizontal interface can also move in the upstream direction. Finally, in the case of waves on the free surface of a vertical liquid film, horse-shoe configurations can be steady (dissipation is compensated by pumping), while the decay of disturbances in a horizontal channel cannot be avoided in principle.

Conclusions. A model evolutionary integrodifferential equation is derived for moderately long waves with a small but finite amplitude, which propagate at an arbitrary angle to the steady flow vector. The method proposed can be used not only for stratified Poiseuille flow but also for other flow profiles. Transformation of nonlinear solitary two-dimensional and three-dimensional disturbances is studied numerically with allowance for unsteady shear stresses. The effect of the steady flow velocity and direction on the wave amplitude and shape is demonstrated.

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